Demonstrating Passivity and Dissipativity using Computational Methods

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Abstract

Passivity and dissipativity are energy based properties of dynamical systems that may be used for the analysis and synthesis of linear and nonlinear systems. The two properties provide valuable stability results as well as compositional results for the analysis of interconnected systems. Using both the stability and compositionality results, large scale systems can be determined to be stable by analyzing the components in terms of energy dissipation and then sequentially analyzing the system interconnections. One of the drawbacks of this approach is that demonstrating that a system is passive or dissipative typically requires finding an energy storage function, which is analogous to a Lyapunov function. As with Lyapunov stability, the search for a storage function to show dissipativity is in general an open-ended search.

This paper surveys computational methods for finding energy storage functions. This includes linear matrix inequality (LMI) methods for linear systems and sum of squares (SOS) methods for polynomial nonlinear systems. When these methods are applicable, the search for storage functions can be automated to greatly simplify analysis and synthesis of linear and nonlinear systems. New material is provided on the application of these methods to find passivity indices for dynamical systems. Additional material is provided on using SOS methods to demonstrate dissipativity for switched systems. Examples are provided to illustrate how these methods may be used in practice.

1　Introduction

Dissipativity is an energy based property of dynamical systems [1, 2]. A system is dissipative if it only stores and dissipates energy provided by the environment without generating its own energy. The energy stored in the system is captured by an energy storage function. The rate energy is supplied by the environment can be specified by an energy supply rate. This supply rate can be chosen to capture system properties including passivity and $\mathcal{L}_2$ stability. While general dissipativity allows for the most general results, the special case of QSR dissipativity is more relevant for computational results [3, 4]. The special case of passivity provides valuable results for analysis of dynamical systems. The notion of passivity is based on electrical circuit analysis where circuits made up of passive components were known to be stable and form stable feedback loops [5, 6]. Passive systems are Lyapunov stable and the property of passivity is preserved when systems are combined in feedback or parallel [7]. These facts can be used to design stabilizing controllers for passive systems. In our previous work, these methods have been valuable for analyzing network control systems [8, 9, 10, 11].

The results that follow from dissipativity are only valuable as long as it is possible to show that a given system is dissipative. Demonstrating dissipativity requires choosing a particular energy supply rate, and then finding an energy storage function. Finding an energy storage function is analogous to finding a Lyapunov function when demonstrating stability, which is a challenging problem in general. The results provided by dissipativity are much more valuable when the process of showing the property can be automated. When dissipativity can be shown computationally, the analysis and synthesis involved in control system design is greatly simplified. Traditionally this was only done for linear systems with quadratic supply rates using Linear Matrix Inequalities (LMIs) [12]. Recently, this has been extended to a class of nonlinear systems that are modeled using polynomial equations. Dissipative polynomial systems can be shown to be so with a polynomial energy storage function by using Sum of Squares (SOS) methods [13].

An important special case of dissipativity is the passivity index framework where the energy supply rate is characterized by two indices. The passivity indices directly generalize the notion of passivity by characterizing the level of passivity present in the system. The concept of indices was defined in [14, 15] and was based on the notion of conic systems [16]. Thorough background on this topic can be found in [7, 17]. Previously, no work has been published on computational methods of determining passivity indices using computational methods such as SOS optimization. Some initial work in this direction is provided in this report.

The notions of passivity and dissipativity have been extended to switched systems in continuous [18, 19] and discrete time [20, 21, 22]. Additionally, passivity indices have been defined for switched systems [23]. The definitions in these papers use multiple storage functions to show dissipativity. This is based on the application of Lyapunov stability to switched systems using multiple Lyapunov functions [24, 25]. For dissipativity, multiple storage functions capture the fact that energy may be stored differently for each mode of the system. While these papers contain valuable results, the burden of demonstrating dissipativity is increased due to the requirement of finding several storage functions. This report explores SOS methods to facilitate using dissipativity for switched systems.

This paper is organized into the following sections. Section 2 surveys existing methods of demonstrating dissipativity and passivity for linear systems using LMIs. Section 3 covers demonstrating dissipativity for nonlinear polynomial systems. This includes previous work on passivity for polynomial systems and new work on dissipativity for polynomial systems. Section 4 covers the

special case of passivity indices where SOS methods may be used to computationally determine the indices. Section 5 covers the extension of these methods to switched systems. Examples are provided throughout the paper to illustrate how these methods can be applied. Concluding remarks are given in Section 6.

2   LMIs for Linear Systems

Linear time-invariant (LTI) systems represent an important class of dynamical systems. This class includes many practical examples from classical control. Indeed, all Lipschitz continuous dynamical systems can be accurately modeled as a linear system when operating within some neighborhood of a desired equilibrium point [26].

The continuous time LTI systems of interest may be written,

$$
\begin{aligned}
\dot{x} &= Ax + Bu \\
y &= Cx + Du.
\end{aligned}
\tag{1}
$$

The matrices A, B, C, and D are of appropriate dimension defined by vectors $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $y \in \mathbb{R}^p$. It is well known that unforced linear systems $(u(t) = 0, \forall t)$ are Lyapunov stable if the matrix A is Hurwitz, i.e. the eigenvalues of A are in the closed left half plane [26].

This paper will focus on continuous time systems but notes are provided for the discrete time case. Discrete time LTI systems can be written,

$$
\begin{aligned}
x(k+1) &= Ax(k) + Bu(k) \\
y(k) &= Cx(k) + Du(k).
\end{aligned}
\tag{2}
$$

2.1   Background on LMI Methods

When working with linear systems, many system properties can be formulated as linear inequalities with unknowns. When appropriate parameters can be found to satisfy the inequalities, the system has the desired property. The search for the unknown parameters can often be formulated as a semi-definite optimization problem so that computational solvers may be used to verify the desired property. In control systems analysis, LMIs can be used to determine stability, optimality, robustness, and other properties. A thorough summary of control problems that may be solved using LMIs may be found in [12].

An example of a problem that can be solved using LMI methods is Lyapunov stability of an unforced linear system. It is well established that linear systems are stable if and only if there exists a quadratic supply rate $V(x) = x^T P x$ such that $V > 0$ and $\dot{V} \leq 0$ [27]. The function V being positive is satisfied by the LMI $P > 0$. Ensuring that the derivative of V is negative semi-definite can be shown in the following.

$$
\begin{aligned}
\dot{V}(x) &= \dot{x} P x + x^T P \dot{x} \\
&= (Ax)^T P x + x^T P (Ax) \\
&= x^T (A^T P + PA) x \\
&\leq 0
\end{aligned}
$$

Clearly this result holds when the matrix $A^T P + PA$ is negative semi-definite. We can test for stability by searching for a matrix P that simultaneously satisfies $P > 0$ and $A^T P + PA \leq 0$. This search is an LMI feasibility problem. When feasible, the linear system is stable. Since this result is necessary and sufficient, when the LMI problem can be shown to be infeasible, the lack of a solution implies that the linear system is unstable.

The general LMI optimization problem can be written,

$$\min c^T x$$
$$\text{subject to } Ax \leq b.$$

This problem can be solved using solvers such as `mincx` provided by the Robust Control Toolbox in `MATLAB`. For many problems the quantity to minimize is arbitrary. Instead finding just a single feasible solution solves the problem of interest. This type of problem can be solved using `feasp` in the Robust Control Toolbox. These solvers make use of interior point methods from linear programming to produce solutions in polynomial time [28].

2.2 LMIs to Show Passivity

The property of passivity originated in the study of electronic circuits made up of passive components. It was known that passive circuits are stable and that interconnecting two passive circuits in feedback forms a new circuit that was both passive and stable. There is no such stability guarantee when connecting stable circuits in feedback. This simplified the design of stable circuits [5, 6]. Although this theory was well known for circuits, the results are not dependent on the supply rate (power) being in a standard unit of energy such as Watts or the energy storage function capturing electrical energy. Passivity can be applied to linear and nonlinear systems with a generalized notion of energy quantified by an energy storage function. The property was defined for a state space representations in [2].

The key properties from passive circuit analysis carried over to state space systems. For one, these systems are Lyapunov stable. Additionally, the feedback interconnection of two passive systems remains passive (Fig. 1). These two facts together imply that a feedback loop made up of any two passive systems is Lyapunov stable. Essentially, passivity provides open loop conditions
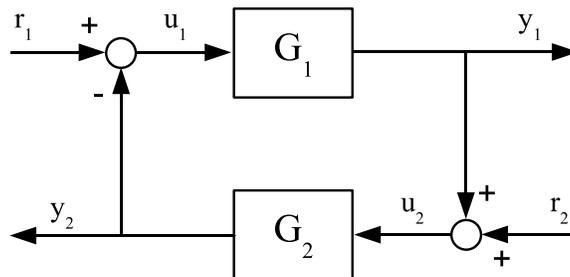
Figure 1: The feedback interconnection of two dynamical systems G_1 and G_2.

for closed loop stability. Additionally, passivity is preserved when two systems are combined in parallel. Large scale systems that are stable can be designed by systematically connecting passive components in feedback or parallel. For more detail on passivity theory, refer to [27, 7, 29].

4

Definition 1. *[2, 27, 7] A dynamical system is passive with respect to an input space $U \subset \mathbb{R}^m$ if there exists a non-negative energy storage function $V(x)$ such that the energy stored in the system is bounded above by the energy supplied $\int u^T y dt$ to the system over any finite time interval, i.e. for $u(t) \in U$ and $\forall t_1, t_2$ s.t. $t_1 \leq t_2$*

$$\int_{t_1}^{t_2} u^T y dt \geq V(x(t_2)) - V(x(t_1)). \tag{3}$$

This definition is with respect to an input set U. For the purposes of this paper it will be assumed that $U = \mathbb{R}^m$. Note that this definition of passivity is standard for linear as well as nonlinear systems. Passive linear time-invariant (LTI) systems are also known as positive real systems.

The use of LMIs in passivity theory comes from earlier work on the positive-real lemma, also known as the KYP Lemma. This lemma was developed by Kalman [30] using results from Yakubovich [31, 32] and Popov [33]. The lemma has been extended to nonlinear systems [29].

Consider a passive linear system (A, B, C, D) with quadratic storage function, $V(x) = \frac{1}{2}x^T P x$, where P is positive definite. We can limit our attention to quadratic storage functions without loss of generality because a linear system is passive if and only if there exists a quadratic energy storage function [27]. Analyzing the derivative of the energy storage function and comparing it to the passive inequality yields the following derivation.

$$\dot{V}(x) = \frac{1}{2}\left[\dot{x}Px + x^T P\dot{x}\right]$$

$$= \frac{1}{2}\left[(Ax + Bu)^T Px + x^T P(Ax + Bu)\right]$$

$$= \frac{1}{2}\left[x^T(A^T P + PA)x + x^T PBu + u^T B^T Px\right]$$

$$\leq u^T y = \frac{1}{2}\left[u^T y + y^T u\right]$$

$$= \frac{1}{2}\left[u^T Cx + u^T Du + x^T C^T u + u^T D^T u\right]$$

$$\Longleftrightarrow \begin{bmatrix} x \\ u \end{bmatrix}^T \begin{bmatrix} A^T P + PA & PB - C^T \\ B^T P - C & -D - D^T \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}^T \leq 0 \tag{4}$$

This derivation shows that a system is passive when the matrix on the last line is negative semi-definite. The problem of finding such a P to make this matrix negative can be formulated as an LMI. Assuming that the LMI problem is feasible, such a P exists (and thus $V(x)$) to show passivity. This test is a necessary and sufficient test to show whether or not an LTI system is passive. When the test fails, it can be concluded that the system is not passive.

An example is provided to illustrate the LMI methods for passive LTI systems. As a quick note, the system considered satisfies the necessary conditions of passivity, i.e the system is stable, minimum phase, and has relative degree zero. However, these conditions are not sufficient to conclude passivity so an LMI is used to determine passivity. The solver `feasp` in `MATLAB` is used to solve this feasibility problem.

Example 1. *Consider a continuous time linear system,*

$$
\begin{aligned}
\dot{x} &= \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\
y &= \begin{bmatrix} -1 & 2 \end{bmatrix} x + 1.5u.
\end{aligned}
\tag{5}
$$

Passivity of this system can be tested using LMIs by assuming a storage function of the form $V(x) = \frac{1}{2} x^T P x$. *The solver provides a solution,*

$$
P = \begin{bmatrix} 5.59 & 0.96 \\ 0.96 & 1.94 \end{bmatrix}.
\tag{6}
$$

The matrix P is positive definite since both eigenvalues are positive: 1.698 *and* 5.824. *Passivity can be verified by substituting the matrix P into the LMI (4). The LMI evaluates to*

$$
\begin{bmatrix}
-3.84 & -0.203 & 1.96 \\
-0.203 & -5.82 & -0.065 \\
1.96 & -0.065 & -3.00
\end{bmatrix},
\tag{7}
$$

which is negative definite since all eigenvalues are less than zero: -5.860, -5.392, *and* -1.408.

The methods in this section to show passivity for continuous time systems can also be applied to discrete time systems with dynamics (2). It is assumed that the storage function is quadratic, $V(x) = \frac{1}{2} x^T P x$. The LMI test for passivity can be written,

$$
\begin{bmatrix} x \\ u \end{bmatrix}^T
\begin{bmatrix} A^T P A - P & A^T P B - C^T \\ B^T P A - C & -D - D^T \end{bmatrix}
\begin{bmatrix} x \\ u \end{bmatrix}^T \leq 0
\tag{8}
$$

For more on LMIs for passive linear systems, refer to the following surveys and the references therein. [34, 35].

2.3   LMIs to Show Dissipativity

Dissipativity is a property of dynamical systems that is more general than the notions of passivity and $\mathcal{L}_2$ gain for state space systems. This property provides valuable tools for analysis and control of dynamical systems. Unlike passivity, dissipative systems aren't necessarily stable and don't always always form stable feedback loops. However, dissipativity theory does provide sufficient conditions to assess stability for individual systems and for feedback loops. These results can be applied to cases when other sufficient results such as the passivity theorem or the small gain theorem fail. This generalization is accomplished by varying the notion of energy supplied to the system. When an energy supply rate is chosen for a particular system, the actual dynamics of the system are abstracted to this dissipative property which may be used to assess stability or other desired properties.

There are important control synthesis methods that follow from the property of dissipativity. This is done by first finding a dissipative rate (or a class of rates) that is valid for a given plant. This rate can be used to find a set of complementary dissipative rates for a controller. These complementary dissipative rates guide the design of the controller by providing bounds on the allowable dissipative behavior. When these guidelines are followed, stability of the feedback connection of plant and controller is guaranteed.

Definition 2. *[1, 7, 29] A system is dissipative with respect to an input space $U \subset \mathbb{R}^m$ if there exists a non-negative energy storage function $V(x)$ such that the energy stored in the system is always bounded above by the energy supplied $\omega(u, y)$ to the system over any finite time interval, i.e. for $u(t) \in U$ and $\forall t_1, t_2$ s.t. $t_1 \leq t_2$*

$$\int_{t_1}^{t_2} \omega(u, y) dt \geq V(x(t_2)) - V(x(t_1)). \tag{9}$$

For this paper, the input space will be assumed to be, $U = \mathbb{R}^m$.

It should be noted that it isn't possible to use LMIs to demonstrate dissipativity for all supply rates ω since there is no guarantee that the resulting inequality will be linear in the unknown terms. When we impose a quadratic structure on the dissipative rate, it is possible to assess a variety of dissipative behaviors by using LMIs. The following quadratic supply rate comes from the work of Hill and Moylan [3, 4].

Definition 3. *[3] A system is QSR-dissipative if it is dissipative with respect to the following supply rate,*

$$\omega(u, y) = \begin{bmatrix} y \\ u \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix}, \tag{10}$$

where $Q = Q^T$ and $R = R^T$.

The parameters Q, S, and R can be chosen to assess a variety of dissipative behaviors including passivity and finite-gain $\mathcal{L}_2$ stability. In general, dissipativity can be used to assess other behavior such as asymptotic stability or input-to-state (ISS) stability [34].

As in passivity theory, it is possible to use LMIs to determine whether or not a system is QSR dissipative for a fixed Q, S, and R. The storage function is again assumed to be quadratic, $V(x) = \frac{1}{2} x^T P x$, with the constraint $P > 0$. The following LMI can be derived by generalizing the positive real lemma.

$$\begin{bmatrix} A^T P + PA - C^T QC & PB - C^T S - C^T QD \\ B^T P - S^T C - D^T QC & -D^T QD - S^T D - D^T S - R \end{bmatrix} \leq 0 \tag{11}$$

This LMI along with the condition that $P > 0$ represent the constraints that the optimization problem is subject to. As with passivity, the **MATLAB** solvers can be used to find an appropriate P to satisfy the LMIs and complete the storage function. The discrete time version of this LMI can be written,

$$\begin{bmatrix} A^T PA - P - C^T QC & A^T PB - C^T S - C^T QD \\ B^T PA - S^T C - D^T QC & -D^T QD - S^T D - D^T S - R \end{bmatrix} \leq 0. \tag{12}$$

More details about these LMIs can be found in [35].

3 SOS for Nonlinear Systems

This section focuses on SOS methods to show passivity and dissipativity for nonlinear systems. The nonlinear systems of interest are ones with only polynomial terms. Examples of polynomial systems can be found in applications such as biological systems and process control. A detailed list of common polynomial nonlinearities can be found in [13].

The polynomial systems of interest have the following form,

$$\begin{aligned}\dot{x} &= f(x) + g(x)u \\ y &= h(x) + j(x)u,\end{aligned} \tag{13}$$

where f, g, h, and j are polynomial functions of the state and of appropriate dimension given by $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $y \in \mathbb{R}^p$. It should be noted that linear systems are a special case of polynomial systems. We know that nonlinear systems are stable when there exists a positive Lyapunov function $V(x)$ such that $\dot{V} \leq 0$. When the Lyapunov function is a polynomial function of the state, it is referred to as a polynomial Lyapunov function.

SOS methods can only be directly applied to polynomial systems. However, many nonlinear terms that are not polynomial can be approximated by a polynomial model, e.g. a finite truncation of the Taylor series expansion of the term. The approximation can be arbitrarily accurate for a given operating region if a high order truncation is used. Alternatively, the polynomial portion of the model dynamics may be handled using SOS methods and any other dynamics that are sector-bounded may be treated using existing methods [27, 7].

3.1 Background on SOS Methods

Several problems in control systems can be formulated as a search for a positive definite (PD) or positive semi-definite (PSD) function, $F(x) \geq 0$, $\forall x \in \mathbb{R}^n$. It is clear that this is the case for Lyapunov stability, by showing that V is PD and $-\dot{V}$ is PSD the system is stable. This is also the case for other problems in control systems such as Lyapunov stability [36], robustness [37], region of attraction [37], hybrid system verification [38, 39], stability with delays [40] and several others.

Traditionally, these problems are computationally efficient to solve for linear systems using LMIs while they are not computationally feasible in the nonlinear case. In general, these problems are non-convex and np-hard [36]. These problems have recently been approached for polynomial nonlinear systems using semi-definite programming. The key step in formulating the optimization problem is in replacing the positive semi-definite condition with an alternative sufficient condition. This condition is to show that the function is instead a sum of squares (SOS) of lower order polynomials [36].

$$F(x) = \sum_i f_i^2(x) \geq 0 \tag{14}$$

Clearly, a function being SOS implies that the function is PSD although it is not a necessary condition in general.

We consider polynomials that are functions of an nth dimensional variable. In order for a polynomial to be a sum of squares, the degree m must be even. We take the function of interest $F(x)$ and write it in the following form,

$$F(x) = z^T(x)Qz(x), \tag{15}$$

where z is the stacked vector of all possible monomials up to degree $m/2$. For example, considering the case when $n = 2$ (pick variables x_1 and x_2) and $m = 4$ the possible monomials are given by,

$$z(x) = \begin{bmatrix} x_1 \\ x_2 \\ x_1^2 \\ x_1 x_2 \\ x_2^2 \end{bmatrix}. \tag{16}$$

The matrix form is known as a Gram matrix representation. It has been shown that $F(x)$ has a sum of squares decomposition if and only if it can be written as in (15) with a positive semidefinite Q [41]. This result enabled the use of semi-definite programming for polynomial nonlinear systems. An example is given to demonstrate the matrix decomposition.

Example 2. *Consider the polynomial*

$$f(x) = 2x_1^2 + x_1^4 - 2x_1^3 x_2 + 6x_1^2 x_2^2 - 6x_1 x_2^3 + 9x_2^4. \tag{17}$$

This polynomial may be written in Gram matrix form (15)

$$f(x) = \begin{bmatrix} x_1 \\ x_1^2 \\ x_1 x_2 \\ x_2^2 \end{bmatrix}^T \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 6 & -3 \\ 0 & 0 & -3 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_1^2 \\ x_1 x_2 \\ x_2^2 \end{bmatrix}. \tag{18}$$

where the matrix Q is positive definite. This guarantees that $f(x)$ can be written as a SOS,

$$f(x) = 2x_1^2 + (x_1^2 - x_1 x_2 + 3x_2^2)^2. \tag{19}$$

It should be noted that the matrix Q given in the example is not unique. It is possible to characterize all such matrices that can represent $f(x)$ in this form. This is done by identifying the matrix N such that $z^T(x)Nz(x) = 0$. Any matrix $Q + \lambda N$ for real valued λ can also represent $f(x)$. Finding the Gram matrix decomposition does not automatically provide the SOS representation. When an appropriate matrix factorization of Q can be found, it is possible to determine the SOS representation of the function [42].

A typical control problem using SOS methods begins with a given system with polynomial dynamics and assumes a polynomial form for an unknown function. This class of problems can be setup as a semi-definite optimization problem. When the problem is feasible, there exists a function that is SOS which implies that it is PSD. The optimization problem can be written in the following way.

$$
\begin{array}{ll}
\min & c_1 u_1 + ... + c_n u_n \\
\text{subject to} & P_i(x) = A_{i,0}(x) + A_{i,1}(x)u_1 + ... + A_{i,n}(x)u_n \\
\text{for} & i = 1, ..., n
\end{array} \tag{20}
$$

This problem can be shown to be convex and solvable using semi-definite programming. Computational solvers are available such as SOSTOOL [43] for **MATLAB**. SOSTOOL relies on semi-definite solvers such as SeDuMi.

3.2 SOS to Show Passivity

Traditional passivity theory requires finding a non-negative storage function $V(x)$ such that $\dot{V} \leq u^T y$. This problem can be rephrased as finding parameters such that V and $u^T y - \dot{V}$ that are PSD. Developing a computational method of demonstrating passivity for polynomial systems mirrors the work of Lyapunov stability for polynomial systems. The original contribution was in recognizing that it is possible to relax these conditions to other sufficient conditions. In this case, instead of showing that V and $u^T y - \dot{V}$ are PD, the problem is altered to show that they are SOS. This fact has been pointed out recently [34].

Once the problem is formulated this way, there are efficient solvers to find a solution. Like the other SOS problems, the existence of a SOS storage function is only sufficient for showing passivity. In general, it isn't possible to use SOS methods to demonstrate that a system is not passive. Other problems related to passivity have been approached using SOS. One such problem is the search for state feedback to render a system passive [44, 45].

It may help to illustrate how SOS methods can be used with an example. The following demonstrates how SOS methods can be used to show passivity of a polynomial nonlinear system.

Example 3. *Consider a polynomial nonlinear system,*

$$
\begin{aligned}
\dot{x} &= \begin{bmatrix} -(x_1^3 + x_1 x_3^2)(1 + x_3^2) \\ -(x_1^2 x_2 + x_2)(1 + x_3^2) \\ -(x_3 + x_1^2 x_3)(1 + x_3^2) - 3x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \\
y &= \begin{bmatrix} x_3 \end{bmatrix}.
\end{aligned}
\tag{21}
$$

*A storage function is chosen to be of the form $V(x) = x^T P x + a_1 x_1^4 + a_2 x_2^4 + a_3 x_3^4$. Using SOSTOOLS in **MATLAB** the system is shown to be passive with $P = diag\{1.70, 1.29, 0.5\}$, $a_1 = 0.867$, $a_2 = 0.815$, and $a_3 = 0$ which yields the storage function $V(x) = 1.70x1^2 + 1.29x2^2 + 0.5x3^2 + 0.867x1^4 + 0.815x2^4$. A quick check can be done to verify passivity by evaluating $\dot{V}(x) = \frac{\partial V(x)}{\partial x} [f(x) + g(x)u]$ which will be bounded above by $y^T u$.*

It is important to remember that the relaxation of the problem to search for functions that are SOS instead of positive is a sufficient only test of passivity. When the optimization problem fails to find parameters of the storage function to show passivity, it cannot be concluded that the system is not passive. Instead, the test is inconclusive.

3.3 SOS to Show Dissipativity

The same sort of derivation applies to the search for storage functions for dissipative polynomial systems. Once again, general dissipativity does not immediately allow a computational algorithm to be formulated to find a storage function. However, we don't have to restrict ourselves to the QSR-dissipative rate. Whenever the suply rate ω is a polynomial function of the arguments u and y, a semi-definite optimization problem can be formulated to search for a storage function. At this point, the feasibility problem can be run to find an energy storage function $V \geq 0$ such that,

$$
\omega(u, y) - \dot{V} \geq 0.
\tag{22}
$$

As before, the algorithm relaxes the positive conditions to instead find parameters that guarantee that V and $\omega(u, y) - \dot{V}$ are SOS.

SOS optimization methods have been employed to find energy storage functions for dissipative polynomial systems for specific energy supply rates [13, 46, 47]. These papers show that dissipation inequalities involving an unknown storage function can be formulated as a SOS problem. The authors investigate dissipative inequalities for the minimum phase property, robustness, and synchronizing feedbacks. In each case, they derive a new inequality that can be solved using SOS.

4 Computational Methods for Passivity Indices

The passivity index framework represents an important intermediate analysis method between passivity and dissipativity. The framework is a special case of dissipativity where the dissipative rate is characterized by two parameters: the passivity indices. The indices generalize passivity to apply feedback stability results to systems that may not be passive. While passivity can only be directly used to assess systems as passive or not passive, passivity indices capture the level of passivity in a system. This section will introduce the indices, cover stability results, and discuss how SOS methods may be applied. Examples using SOS methods to find passivity indices are provided. More details on passivity indices may be found in [7, 17].

4.1 Background on Passivity Indices

The level of passivity of a system can be captured by using the two passivity indices. They are defined so that a positive value indicates an excess of passivity and a negative value indicates a shortage of passivity. A passive system must have both indices positive or zero.

The first index is the output feedback passivity (OFP) index. This is a measure of the level of stability of a system. It is defined as the largest gain that can be placed in positive feedback that still forces a system to be passive (Fig. 2). For unstable systems, this value will be negative.

Definition 4. *The output feedback passivity index (OFP) is the largest gain that can be placed in positive feedback with a system such that the interconnected system is passive (Fig. 2). This notion is equivalent to the following dissipative inequality holding for the largest ρ,*

$$\int_0^T u^T y \, dt \geq V(x(T)) - V(x(0)) + \rho \int_0^T y^T y \, dt, \; \forall T. \tag{23}$$
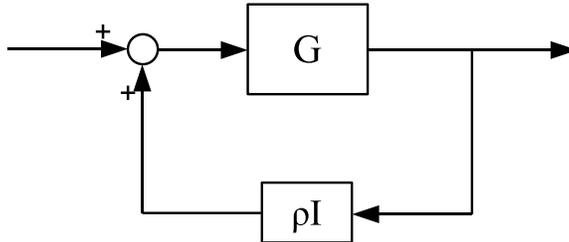
Figure 2: This block diagram demonstrates the physical significance of the OFP index ρ that is the largest gain that compensates for an excess or shortage of stability to exactly passivate G.

The second index is the input feedforward passivity (IFP) index. It is dependent on the feedforward term $j(x)$ of a system (in eq. (13)) and the stability of the zero dynamics. A positive value indicates that the term $j(x)$ is positive for all x and that the zero dynamics are asymptotically stable. If $j(x) < 0$ for some x or the zero dynamics are unstable, the index will be negative.

Definition 5. *The input feedforward passivity (IFP) index is the largest gain that can be put in a negative parallel interconnection with a system such that the interconnected system is passive (Fig. 3). This notion is equivalent to the following dissipative inequality holding for the largest ν,*

$$\int_0^T u^T y \, dt \geq V(x(T)) - V(x(0)) + \nu \int_0^T u^T u \, dt, \; \forall T. \tag{24}$$

11
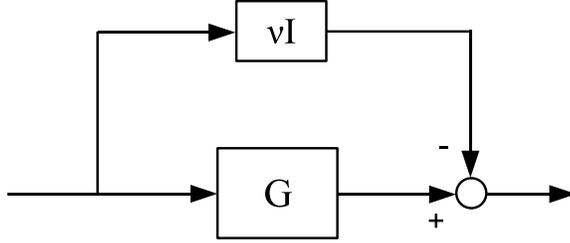
Figure 3: This block diagram shows that the IFP index is the largest feedforward gain ν that compensates for an excess or shortage of the minimum phase property of system G.

These definitions of passivity indices consider only one index at a time. When only one index is considered, the other is implicitly considered to be zero. Often it is worth considering non-zero values for both indices, and these simultaneous indices can be visualized in Fig. 4. Applying both indices is the equivalent to the system being dissipative with respect to the energy supply rate,

$$\omega(u, y) = (1 + \rho\nu)u^T y - \rho y^T y - \nu u^T u. \tag{25}$$
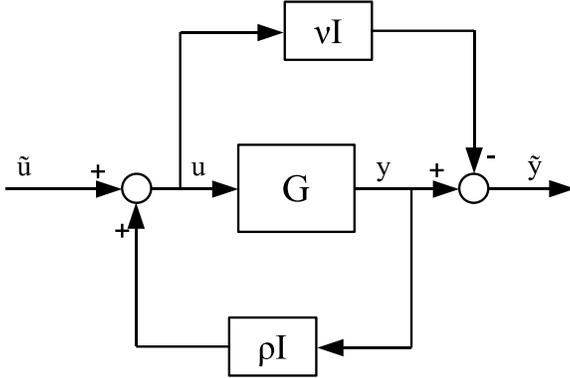
Figure 4: This block diagram demonstrates the definition of the indices when both are applied.

It should be noted that this dissipative rate is not simply the sum of the two terms from the individual definitions of ρ and ν. This dissipative rate allows a strong connection to be made to conic systems [16] and allows for a block diagram to be drawn that provides a physical interpretation for the indices. The connection to conic systems was explored in [48].

As stated earlier, the main reason to use passivity indices is to extend the feedback stability results provided by passivity theory. It is well established that the feedback interconnection (Fig. 1) of two passive systems is passive and stable. If one of the systems is not passive but "almost" passive, the results may not be applied at all. The following result shows how knowing the passivity indices of two systems in feedback may be used to assess stability

Theorem 1. *([17]) Consider the feedback interconnection (Fig. 1) of two nonlinear systems G_1 and G_2 where G_1 has indices (ρ_1,ν_1) and G_2 has indices (ρ_2,ν_2). The interconnection is $\mathcal{L}_2$ stable if the following matrix is positive definite:*

$$A = \begin{bmatrix} (\rho_1 + \nu_2)I & \frac{1}{2}(\rho_1\nu_1 - \rho_2\nu_2)I \\ \frac{1}{2}(\rho_1\nu_1 - \rho_2\nu_2)I & (\rho_2 + \nu_1)I \end{bmatrix} > 0 \tag{26}$$

This is the most general result that can be applied to a large class of nonlinear systems to show $\mathcal{L}_2$ stability. There are less conservative results for LTI systems [48].

4.2 SOS Methods for Passivity Indices

In the basic case, SOS methods can be used to verify that a system has a given pair of passivity indices. This is a special case of the methods for dissipative systems discussed previously. The dissipative rate is chosen to be

$$\omega(u,y) = (1+\rho\nu)u^T y - \rho y^T y - \nu u^T u, \tag{27}$$

for a specific (ρ,ν) pair. At this point, the SOS methods for dissipative systems can be directly applied to verify that the indices hold. It is important to note that this test is sufficient only. In the event that the test fails, this does not indicate that these indices are not valid for the system but only that the test may not be used to assess these indices.

The more interesting case is when SOS methods can be used to find indices or determine maximum indices. Finding the maximum value of an index for a given system is valuable since the maximum provides the least restrictive results to show stability.

For now, consider the case when only a single index is of interest. This may occur when it can be observed that one index has a maximum value of zero. This is the case when a system lacks a feedforward term. For the model (13), when $j(x) = 0$ the index ν has a maximum value of zero. The index may be less than zero if the zero dynamics of the system are unstable. When this occurs, the SOS optimization problem will not successfully terminate. For now, assume that the zero dynamics are stable.

To maximize the index ρ the following optimization problem may be defined.

$$
\begin{aligned}
\min \quad & -\rho \\
\text{subject to} \quad & V(x) = \sum_i a_i p_i^2(x) \\
& u^T y - \rho y^T y - \frac{\partial V}{\partial x} f(x,u) = \sum_i b_i p_i^2(x)
\end{aligned} \tag{28}
$$

The functions $p_i(x)$ are lower order polynomials. If the optimization problem successfully terminates, the parameters of the storage function V are found, and the index ρ is maximized. This method can be seen in the following example.

Example 4. *Consider a system with dynamics*

$$
\begin{aligned}
\dot{x} &= \begin{bmatrix} -2x_1 + x_2 - 0.5x_1^3 \\ -0.5x_1 - x_2 - x_2^3 + u \end{bmatrix} \\
y &= x_2 + 2u.
\end{aligned} \tag{29}
$$

Using the SOS method above to maximize ρ, an optimization problem can be set up in SOSTOOLS. The program terminates successfully with storage function

$$V(x) = 0.33x_1^4 + 0.23x_2^4 + 0.31x_1^2 + 0.64x_2^2 \tag{30}$$

and OFP index $\rho = 0.357$. This result can be verified by hand to show that $\frac{\partial V}{\partial x}[f(x) + g(x)u] \leq u^T y - 0.357y^T y$.

This process can be repeated to find the IFP index ν. The optimization problem (28) can be revised to minimize $-\nu$ with the appropriate changes to the constraints.

Example 5. *Reconsidering system (29), it is desirable to estimate the IFP index ν. A similar optimization problem is set up in SOSTOOLS. The program terminates successfully with storage function*

$$V(x) = 0.67x_1^4 + 0.49x_1^2 + 0.50x_2^2 \tag{31}$$

and IFP index $\nu = 2$. This example demonstrates nu has a strong connection to the feedforward term $j(x)$. For many examples, when the zero dynamics are stable and the term $j(x)$ is constant, the IFP index is equal to $j(x)$.

Ideally, an optimization problem would be set up to simultaneously find both ρ and ν. Unfortunately these methods cannot be applied directly to find both indices because the dissipative rate (27) contains the product of the indices. The semidefinite solvers that find solutions to SOS problems can only be solved when the constraints are linear in the unknown variables.

When it is desirable to find both indices, one method of computing such a pair is to fix one index and then maximize the other index. This process can be repeated for several fixed values of ρ and then several fixed values of ν. This method gives several pairs of indices. The "best" pair may be determined by the control designer for the particular application.

Alternatively, the values for the indices can be found by iterating over a range of ρ and a range of ν. The drawback of this approach is that it takes a lot of computational resources to iterate over a large range with fine precision. While this method is typically not efficient, it is possible to speed up the computation by first finding the maximum of each index when the other is set to zero. This narrows the search range to focus the problem and save computations. The computational load can be further reduced by an adaptive optimization problem that first computes pairs of indices on a sparse grid and then refines the grid until the desired level of precision is achieved.

As a side note, the problem of finding an optimal pair of passivity indices is more complicated than properly defining an optimization problem. For most systems there does not exist an objective "best" pair of indices. Instead, the maximal pairs of the indices form a function that appears to always be continuous but may not differentiable. It is often possible to reduce one index in order to increase the other. While the SOS optimization problem may not find pairs exactly on the function, they are typically very close.

5 Computational Methods for Switched systems

SOS methods to demonstrate passivity and dissipativity allow these concepts to be applied to systems more readily. This is especially true for switched systems. This notion of dissipativity for switched systems is based on multiple storage functions. A system with M subsystems will have as many as M energy storage functions. There are additional functions to determine in order to demonstrate dissipativity that may be difficult to find. When a system has more than a few discrete modes, the definition becomes overly cumbersome to apply without computational methods.

The following subsection introduces the notion of dissipativity used for switched systems. Then the problem is reformulated so that SOS methods may be used to find the functions of interest. Finally an example is provided to show how this method may be applied in practice.

5.1 Background on Dissipativity for Switched Systems

A general nonlinear switched system has the following form, where $x \in R^n$, $u \in R^m$, and $y \in R^p$,

$$\begin{aligned}
\dot{x} &= f_\sigma(x, u) \\
y &= h_\sigma(x, u).
\end{aligned} \tag{32}$$

The function $\sigma : \mathbb{R}^+ \to 1, ..., M$ is piecewise constant and indicates the index of the current active subsystem. At any given time, $\sigma(t) = i$ for $i \in \{1, ..., M\}$ and the dynamics are nonlinear and time-invariant. There are a finite number of subsystems M, so i is confined to be an integer from 1 to M. The switching in the system captures discrete behavior from an underlying hybrid process. Each subsystem of the switched system represents a discrete mode of that process so the terms *mode* and *subsystem* will be used interchangeably.

It should be noted that the state variable $x(t)$ is continuous at all times including switching instants where it is typically not differentiable. The input $u(t)$ may be discontinuous but is assumed to have a countable number of discontinuities. In addition the input must be locally square integrable. This assumption is made to avoid the case when $u(t)$ approaches infinity in finite time.

The notion of dissipativity for switched systems using multiple storage functions parallels the work on stability of switched systems using multiple Lyapunov functions [24, 25].

Definition 6. *A switched system (32) is dissipative if there exist positive definite storage functions $V_i(x)$, energy supply rates $\omega_i^i(u, y)$ and cross supply rates $\omega_j^i(u, y, x, t)$ such that the following conditions hold.*

1. *Each subsystem i is dissipative with respect to ω_i^i while active, i.e. for $t_{i_k} \leq t_1 \leq t_2 \leq t_{i_k+1}$ and $\forall i, k$,*

$$\int_{t_1}^{t_2} \omega_i^i(u, y)dt \geq V_i(x(t_2)) - V_i(x(t_1)). \tag{33}$$

2. *Each subsystem j is dissipative with respect to ω_j^i when the ith subsystem is active, i.e. $\forall j \neq i$, and for $t_{i_k} \leq t_1 \leq t_2 \leq t_{i_k+1}$,*

$$\int_{t_1}^{t_2} \omega_j^i(u, y, x, t)dt \geq V_j(x(t_2)) - V_j(x(t_1)). \tag{34}$$

3. *For all i and j there exist absolutely integrable functions $\phi_j^i(t)$ and some input $u^*(t)$ that may depend on the state $x(t)$ such that the following three conditions hold, $\forall t \geq t_0$,*

   - $f_i(0, u^*(t)) \equiv 0$,
   - $\omega_i^i(u^*, y) \leq 0$, *and*
   - $\omega_j^i(u^*, y, x, t) \leq \phi_j^i(t), \forall j \neq i$.

This definition can be narrowed to passivity when the supply rate $\omega_i^i(u, y) = u^T y$ for all i. It should be noted that when considering passivity the input $u^*(t) = 0$ trivially satisfies most of the third condition. However, the existence of functions $\phi_j^i(t)$ is still not trivial. With this new definition of passivity, the authors show that passive switched systems with all storage functions satisfying $V_i(0) = 0$ are Lyapunov stable and that the feedback interconnection of two passive systems is again passive.

5.2 SOS for Switched Systems

For SOS methods to be used to show dissipativity, the switched system must have polynomial dynamics. This means that for all modes i the functions $f_i(x, u)$ and $h_i(x, u)$ must be polynomial in x and u. Additionally, all functions involved in showing dissipativity must be polynomial. This includes the energy supply rates ω_i^i, the energy storage functions V_i, and the cross supply rates ω_j^i.

The first step in showing dissipativity for a switched system is in specifying the energy supply rates. It is assumed that these are given for a particular switched system. These are fully specified if passivity is the property of interest. In some other cases these may be parameterized, and the parameters can be found by the semi-definite solver simultaneously with the storage functions. In the general case, these must be specified for each mode of the system in advance.

The next step in showing dissipativity is to find an energy storage function for each mode of the system. These storage functions are dependent on the energy supply rate specified previously (33). A SOS optimization program can be defined to find each storage function assuming that a form is chosen for the storage function. Luckily, the forms for the storage functions can be generated mostly automatically. This can be done when the variables of interest $\{x_1, \cdots, x_n\}$ and the desired order of the storage function are specified. For linear systems, a quadratic form can be chosen, $V_i(x) = x^T P_i x$ where $P_i = P_i^T$. The symmetry of P_i reduces the number of parameters to find from n^2 to $\frac{1}{2}(n^2 + n)$. In the more general case, the storage function can be parameterized by

$$V_i(x) = z^T(x) Q_i z(x) \tag{35}$$

where $Q_i = Q_i^T$. The form is fully specified when $z(x)$ is chosen and the SOS program finds the elements of Q_i. For example in the case when $x \in \mathbb{R}^n$ and the storage function is fourth order,

$$z(x) = \left[x_1, \cdots, x_n, x_1^2, \cdots, x_n^2, x_1 x_2, \cdots, x_1 x_n, x_2 x_3, \cdots, x_{n-1} x_n \right]. \tag{36}$$

The SOS program can be repeated for each mode of the system to find all storage functions or the problem can be combined into a single large SOS program.

In addition to storage functions, showing dissipativity requires finding as many as $M(M - 1)$ different cross supply rates. Like storage functions, a form must be chosen for the cross supply rates and the parameters may be determined by the optimization problem. For two modes i and j, the form for the cross supply rates can be chosen by comparing the dynamics of the two modes and the storage functions V_i and V_j determined in the previous SOS program. Since these are both known quantities, the forms can be specified computationally.

For example, consider the cross supply rate ω_1^2 which captures the rate energy is supplied to mode 1 when mode 2 is active. The rate depends on the energy storage for mode 1 ($V_1(x)$) and it depends on the energy dissipation of mode 2 which includes $V_2(x)$, ω_2^2, and the dynamics of mode 2. The quantity of interest is the difference between the energy being stored for subsystem 2 and for subsystem 1.

$$\omega_2^2(u, h_2(x, u)) - \frac{\partial V_2}{\partial x} f_2(x, u) + \frac{\partial V_1}{\partial x} f_2(x, u) \tag{37}$$

All the functions in this expression are known already so the expression is fully characterized. Additionally, the functions are all polynomial. The resulting polynomial terms can be used as a candidate for the cross supply rates with unknown coefficients. The coefficients can be found by running a SOS program.

This step is more involved than finding storage functions. For one, there are as many as $M(M-1)$ cross supply rates so there are many more functions to determine. Additionally, each function is dependent on more terms so each SOS program may take longer to execute. However, this analysis method is expected to run off line. Once the storage functions and cross supply rates are determined, they may be used directly without running additional SOS programs.

The third condition of dissipativity involves the search for an input $u(t)$ that has certain properties. This input can be applied to the system in order to stabilize it. The conditions have a strong parallel with control Lyapunov functions that are used to find stabilizing inputs. In some cases the existence of such an input is obvious from the dynamics and the cross supply rates. In other cases it may be very difficult to find such an input. For now, there does not appear to be a SOS method to find this input. For particular systems there may be a quantity to optimize over that will guarantee the existence of that input.

5.3   Example

In the last part of this section, the SOS methods discussed will be applied to a nonlinear switched system to show passivity.

Example 6. *A switched system with two modes is considered. Mode 1 has dynamics given by*

$$
\begin{aligned}
\dot{x} &= \begin{bmatrix} -0.6x_1^3 - 2x_1 + 2x_2 \\ -1.2x_1 - 3x_2 + u \end{bmatrix} \\
y &= \begin{bmatrix} x_2 \end{bmatrix},
\end{aligned}
\tag{38}
$$

and mode 2 has dynamics given by

$$
\begin{aligned}
\dot{x} &= \begin{bmatrix} -2x_1^3 + 0.5x_2 \\ -0.6x_1 - 3x_2 - x_2^3 + u \end{bmatrix} \\
y &= \begin{bmatrix} x_2 \end{bmatrix}.
\end{aligned}
\tag{39}
$$

Both modes are passive when active so $\omega_i(u, y) = u^T y$ for $i = 1, 2$. SOS methods can be used to find storage functions to demonstrate passivity when active. The form of storage function is assumed to contain all terms quadratic or quartic in x_1 and x_2. The optimization problem results in the storage functions,

$$
\begin{aligned}
V_1(x) &= 0.3x_1^2 + 0.5x_2^2 \\
V_2(x) &= 0.6x_1^2 + 0.5x_2^2.
\end{aligned}
\tag{40}
$$

Now SOS methods can be used to find cross supply rates. The following form was chosen for the cross supply rates,

$$
\begin{aligned}
\omega_i^j(u, y, x, t) &= u^T y + a_1 x_1^2 + a_2 x_1 x_2 + a_3 x_2^2 + a_4 x_1^4 \\
&\quad + a_5 x_1^3 x_2 + a_6 x_1^2 x_2^2 + a_7 x_1 x_2^3 + a_8 x_2^4.
\end{aligned}
\tag{41}
$$

The optimization problem gives the cross supply rates,

$$
\begin{aligned}
\omega_2^1(u, y, x, t) &= u^T y - 1.2x_1^4 - x_2^4 + 0.0075x_1^2 - 3x_2^2 \\
\omega_1^2(u, y, x, t) &= u^T y - 0.7x_1^4 - 2.4x_1^2 + 1.2x_1 x_2 - 3x_2^2.
\end{aligned}
\tag{42}
$$

Some terms are not present in the final cross supply rates due to the appropriate coefficient given by the optimization problem being sufficiently near zero. For both cross supply rates, the input $u(t) = 0$

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is considered. The rate ω_1^2 can be bounded by $\phi(t) = ||x(t_0)|| \, e^{-t}$ for all $x \in \mathbb{R}^2$. The rate ω_2^1 may not be bounded for all x. Considering $u(t) = 0$, a region in $\mathbb{R}^2$ may be defined where ω_2^1 is positive. A switching rule may be defined to avoid this region. For this example the region is small and can be defined by $||x_1|| < 0.0866$ and $||x_1|| > 20\,||x_2||$ For this region, only mode 2 may be active. With this mild restriction to the switching, the system is passive and stable for zero input.

This process can be repeated as needed to determine passivity, passivity indices, or more general forms of dissipativity. The only restriction is that the energy supply rates and cross supply rates are polynomials that are linear in the decision variables. Since SOS methods are sufficient only, the failure of a test for a specific energy supply rate does not imply that the system is not dissipative with respect to that supply rate. It only implies that the property cannot be shown using SOS methods for the given forms of storage functions and cross supply rates.

6   Conclusions

This report surveys existing computational methods for demonstrating that a system is passive and dissipative. This included LMI methods for showing that LTI systems are passive or QSR dissipative and SOS methods for demonstrating that polynomial nonlinear systems are passive or dissipative. For existing methods, this paper is presented as a tutorial. Examples are provided for using these methods to improve the applicability of the theory.

In addition to surveying existing results, new methods are presented on SOS methods for showing that polynomial switched systems are passive or dissipative. SOS methods can be used to find both storage functions and cross supply rates. These methods can greatly reduce the effort required to demonstrate that a system is dissipative.

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